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A paper should contain a short and clear summary of the new results obtained and the relations in which they stand to results already known. Contributors are requested to bear in mind that, at the present stage of mathematical research, hardly any paper is likely to be so completely original as to be independent of earlier work in the same direction; and that readers are often helped to appreciate the importance of a new investigation by seeing its connection with earlier results.

The principal results of a paper should, when possible, be enunciated separately and explicitly in the form of definite theorems.

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29 MAR 1927

A SYNTHETIC APPLICATION OF THE SYMBOLIC INVARIANT THEORY TO GEOMETRY

By H. W. Turnbull,
Professor of Mathematics, University of St. Andrews, Scotland.

Introduction.

The following account of some geometrical aspects of the invariant theory, is given in the hope that it may encourage further work in what is a really interesting subject. The presentation does not presume to be novel. It involves much that is generally picked up in passing by anyone who adopts the Hamiltonian Methods of Quaternions or the Grassmann Calculus of Extensive Magnitudes.* It utilizes the ordinary symbolic methods of the algebraic invariant theory, which involve a notation, in the writer's opinion, admirably suited for the concise and clear expression of a large class of projective geometrical theorems. Certain results indeed are proved in this way with startling directness, shewing that these methods have a useful place in any full scheme of geometry. But it would of course be foolish to pretend that they are superior to the direct synthetic methods of pure projective geometry. For although every result in this pure geometry has its invariant algebraic expression, and can be proved by invariant algebra, some such proofs savour more of ingenious gymnastics than lucid disclosures.

§ 1. It is unnecessary to introduce co-ordinates, at any rate in the first instance. We need however two sets of symbols, one for points and one for primes. A *prime* is a flat space of $n-1$ dimensions in space of n dimensions (or n-fold space), denoted by S_n.

Let Greek letters α, β, γ ... together with x, y, z, denote points, and let $a, b, c, \ldots \ldots u, v, w$, denote primes. Then the correspondence here suggested between each α and each a is going to provide

* Cf. *Collected Works.* Also A. N. Whitehead: *Universal Algebra* (Cambridge, 1897).

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us with an accurate notation for the principle of duality. The classification which now follows is called *binary, ternary* and *general,* answering to spaces S_1, S_2 and S_{n-1} respectively.

Binary Forms.

If α, β are two points, let $(\alpha\beta)$ denote a magnitude depending on them such that

$$(\alpha\beta) = -(\beta\alpha) \qquad (I_1)$$

whence $$(\alpha\alpha) = -(\alpha\alpha) = 0.$$

Further, if α, β, γ, δ, are four points in line, let

$$(\beta\gamma)(\alpha\delta) + (\gamma\alpha)(\beta\delta) + (\alpha\beta)(\gamma\delta) = 0 \qquad (I_2)$$

identically.

The expression $(\alpha\beta)$ may be regarded as an ordinary number, *i.e.* *scalar* number, and each term of this last identity is a product of two such numbers, so that the commutative law holds

$$(\beta\gamma)(\alpha\delta) = (\alpha\delta)(\beta\gamma).$$

The elements α, β of $(\alpha\beta)$ are *not* numbers; nor is $\alpha\beta$ when written without the brackets. It then denotes the line (straight line), $\alpha\beta$, through the two points α, β. Thus $(\alpha\beta) = 0$ denotes that two points coincide, while $(\alpha\beta) \neq 0$ is the condition for them to define a line on which they lie.

Correlatively, let

$$(ab) = -(ba) \qquad (I_3)$$

$$(bc)(ad) + (ca)(bd) + (ab)(cd) = 0 \qquad (I_4)$$

denote properties of a pencil of four coplanar lines a, b, c, d, with a common point ab; or of four planes with a common line; or of four primes S_{n-1} with a common S_{n-2}, lying in S_n.

Let the four elements α, β, γ, δ or a, b, c, d be called a harmonic range, or pencil, if two of the three terms in (I_2) or (I_4) are equal*, *e.g.*

$$(\beta\gamma)(\alpha\delta) = (\gamma\alpha)(\beta\delta).$$

* This dogmatic definition is inevitable unless we choose to step outside the binary province, as in § 8.

Then we have enough material here to express all projective properties of such a line, or pencil, without reference to points or primes external to the line or pencil. (To avoid awkward alternatives henceforward, the statement will usually refer to points, leaving the correlative to be understood.).

Ternary forms.

§ 2. Next let $(\alpha\beta\gamma)$ denote a magnitude involving three points α, β, γ, such that

$$(\alpha\beta\gamma) = -(\beta\alpha\gamma) = (\beta\gamma\alpha) = -(\gamma\beta\alpha)$$

$$= (\gamma\alpha\beta) = -(\alpha\gamma\beta) \tag{II_1}$$

a property comprised in saying that $(\alpha\beta\gamma)$ *alternates* in its elements α, β, γ. Let

$$(\alpha\beta\gamma) = 0 \tag{II_2}$$

denote that the three points are in line, and $(\alpha\beta\gamma) \neq 0$, that they define a plane in which they lie. Correlatively let (abc) also alternate, and denote a magnitude involving three coplanar lines a, b, c which meet in a point if

$$(abc) = 0. \tag{II_3}$$

A third magnitude $(a\alpha)$ involving a point α and line a is needed, such that

$$(a\alpha) = (\alpha a),$$

also written

$$a_\alpha. \tag{II_4}$$

Let this vanish if the point α lie on the line a.

Further let the following identities hold for all positions of the co-planar points and lines involved:

$$(bcd)(a\alpha) + (cad)(b\alpha) + (abd)(c\alpha) = (abc)(d\alpha), \tag{II_5}$$

$$(\beta\gamma\delta)(\alpha a) + (\gamma\alpha\delta)(\beta a) + (\alpha\beta\delta)(\gamma a) = (\alpha\beta\gamma)(\delta a), \tag{II_6}$$

$$(abc)(\alpha\beta\gamma) = \begin{vmatrix} (a\alpha) & (b\alpha) & (c\alpha) \\ (a\beta) & (b\beta) & (c\beta) \\ (a\gamma) & (b\gamma) & (c\gamma) \end{vmatrix}, \quad \ldots \tag{II_7}$$

this last being a three rowed determinant.

These form a self-contained and consistent basis for a development of projective geometry of the plane, as is indicated by the Fundamental Theorems of the symbolic invariant theory *

Again if the point α is the intersection of lines b, c, we express this best by writing

$$(u\alpha) \equiv \lambda\,(ubc) \qquad \{u\},$$

the symbol $\{u\}$ meaning that for all positions of the point u in the plane, these magnitudes are identical, λ being a numerical non-zero factor. Substituting in II_7 we find after cancelling (abc) on each side,

$$(\alpha\beta\gamma) = \lambda \begin{vmatrix} (b\beta) & (c\beta) \\ (b\gamma) & (c\gamma) \end{vmatrix},$$

and if now we introduce a further magnitude

$$(bc\,.\,\beta\gamma)$$

by allowing the line u and the line $\beta\gamma$ to be the same we obtain

$$(bc\,.\,\beta\gamma) = \begin{vmatrix} (b\beta) & (c\beta) \\ (b\gamma) & (c\gamma) \end{vmatrix}. \qquad \ldots \quad II_9$$

The general case.

§ 3. This is mostly a direct generalization of the preceding ternary case.

A magnitude $(\alpha\beta\gamma \ldots \mu)$ is attached to n points $\alpha, \beta, \gamma \ldots \mu$, which alternates in $\alpha, \beta, \ldots \mu$ and has $n!$ equivalent arrangements:—

$$(\alpha\beta\gamma \ldots \mu) = -(\beta\alpha\gamma \ldots \mu) = \ldots = -(\mu\beta\gamma \ldots \alpha) \ldots \quad III_1$$

We follow Grassmann and call this the *outer product* of n points in S_{n-1}. It vanishes *i e.*,

$$(\alpha\beta\gamma \ldots \mu) = 0. \qquad \ldots \quad III_2$$

if the points lie in a prime.

* *Cf.* Grace and Young: *Algebra of Invariants* (Cambridge, 1903), 246—280.

Correlatively there is a magnitude $(abc \ldots m)$ attached to n primes $a, b, c \ldots m$, which have a common point if

$$(abc \ldots m) = 0. \qquad \ldots \quad \text{III}_3$$

A third magnitude, Grassman's *inner product* involving a point α and prime a is needed such that

$$(a\alpha) = (-)^{n-1}(\alpha a) = (a \mid \alpha) = (\alpha \mid a) = a_\alpha. \qquad \ldots \quad \text{III}_4$$

All these five notations for the inner product have their use, the first pair being suggested by the outer product of the point α and $n-1$ points, $\beta, \gamma, \ldots \mu$ say, which specify the prime a. For suppose

$$(ax) = (\beta\gamma \ldots \mu x) \qquad \{x\},$$

which is the expression for the fact that the prime a is the unique prime through $n-1$ points $\beta, \gamma \ldots \mu$. Then in particular

$$\begin{aligned}(a\alpha) &= (\beta\gamma \ldots \mu\alpha)\\ &= (-)^{n-1}(\alpha\beta\gamma \ldots \mu)\\ &= (-)^{n-1}(\alpha a)\end{aligned}$$

The same result follows correlatively if for example

$$(u\alpha) = (ubc \ldots m) \qquad \{u\}.$$

This explains the second form in III_4.

The identities holding for all positions in S_{n-1} of points and primes involved are:

$$(rbc \ldots m)(a\alpha) + (arc \ldots m)(b\alpha) + \ldots = (abc \ldots m)(r\alpha), \qquad \ldots \quad \text{III}_5$$

$$(\rho\beta\gamma \ldots \mu)(\alpha a) + (\alpha\rho\gamma \ldots \mu)(\beta a) + \ldots = (\alpha\beta\gamma \ldots \mu)(\rho a), \qquad \ldots \quad \text{III}_6$$

$$(abc \ldots m)(\alpha\beta\gamma \ldots \mu) = \begin{vmatrix} (a\alpha) & (b\alpha) & (c\alpha) & \ldots & (m\alpha) \\ (a\beta) & (b\beta) & (c\beta) & \ldots & (m\beta) \\ \ldots & & \ldots & & \ldots \\ (a\mu) & (b\mu) & (c\mu) & \ldots & (m\mu) \end{vmatrix} \qquad \ldots \quad \text{III}_7$$

Here there are n terms in each left-hand series III_5, III_6, and a determinant of order n in III_7.

The fundamental theorems of symbolic invariant theory* shew that the *outer* and *inner products* together with these three identities III_5, III_6, III_7, are sufficient for the development of projective geometry in space of $n-1$ dimensions. It is however expedient to introduce other magnitudes, suggested, as it were, by III_7 for all lower dimensions. Thus

$$(ab.\alpha\beta) = \begin{vmatrix} (a\alpha) & (b\alpha) \\ (a\beta) & (b\beta) \end{vmatrix} \qquad \ldots \quad III_8$$

$$(abc.\alpha\beta\gamma) = \begin{vmatrix} (a\alpha) & (b\alpha) & (c\alpha) \\ (a\beta) & (b\beta) & (c\beta) \\ (a\gamma) & (b\gamma) & (c\gamma) \end{vmatrix} \qquad \ldots \quad III_9$$

... ...

the last of these being

$$(abc\ldots l.\alpha\beta\gamma\ldots\lambda) \qquad \ldots \quad III_{10}$$

involving $n-2$ primes and $n-2$ points.

Connexion with Co-ordinate Geometry.

§ 4. All that has been said may be easily verified if we assume an outer product to be a determinant whose columns are explicitly indicated, and whose rows are furnished by the n homogeneous co-ordinates needed to specify either point or prime in S_{n-1}. Or again $n-1$ Cartesian co-ordinates may furnish the rows if the last row is entirely filled with ones.

For instance if $n = 4$, $(a|x) = 0$ can be considered the same as $(\alpha\beta\gamma x) = 0$, both denoting a plane.

In homogeneous co-ordinates $[x_1, x_2, x_3, x_4]$ we should define $(\alpha\beta\gamma x)$ as

$$\Delta = \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 & x_1 \\ \alpha_2 & \beta_2 & \gamma_2 & x_2 \\ \alpha_3 & \beta_3 & \gamma_3 & x_3 \\ \alpha_4 & \beta_4 & \gamma_4 & x_4 \end{vmatrix},$$

and $(a|x)$ as $\quad a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4,$

* *Cf.* Grace and Young, *loc cit.* Also Weitzenbock : *Invariantentheorie* (Groningen, 1923), Ch. IV.

where a_i is the co-factor of α_i in Δ. This explains why $(a|x)=(x|a)$. The notation (ax) however arises from the determinant directly; in fact here

$$(ax) = (\alpha\beta\gamma x) = -(x\alpha\beta\gamma) = -(xa).$$

I have however presented the properties of this algebra deliberately in a geometrical form, and have kept the co-ordinates, and all that they suggest, in the back-ground, simply because the co-ordinates are unnecessary. This plan is adopted partly in the hope that it may suggest a strict logical enquiry into the preceding assertions without either assuming the use of co-ordinates or utilizing the fundamental theorems of the invariant theory.

Where the above treatment differs from Grassmann's is in such statements as

$$(u\alpha) = (ubc \dots m) \qquad \{u\}$$

which are used rather than

$$\alpha = bc \dots m;$$

for this last lacks the precision afforded by the identity in u expressed in invariant form, *i.e.*, by use of inner and outer products only.

Relevant and Irrelevant Co-ordinates.

§ 5. A geometrical theorem involves points or other elements for its subject matter. Such may yield what may be called a *relevant* or *intrinsic co-ordinate system*, as opposed to a haphazard frame of reference which has no proper connexion with the geometrical figure of the theorem. In such a case the relevant co-ordinates are invariant functions of the magnitudes called inner and outer products, already explained.

Thus the identities I_4, II_5, III_5 provide relevant homogeneous co-ordinates for any theorem involving these elements. So the point x would have co-ordinates

$$[(\beta\gamma x),\ (\gamma\alpha x),\ (\alpha\beta x)] \tag{1}$$

in a plane, referred to a triangle of reference $\alpha\beta\gamma$. *The first factor of each term on the left of* II_5, II_6, III_5, III_6, *is a co-ordinate*, referred to the triangle (or polyhedron) of reference whose vertices or primes are

explicitly given by the first factor of the term on the right. This factor must be non-zero.

Such relevant co-ordinates are homogeneous. It is worth while adjoining to them *absolute co-ordinates*, obtained therefrom by dividing each co-ordinate by this non-zero factor on the right. For the triangle $\alpha\beta\gamma$ the co-planar point would then have absolute co-ordinates

$$\left[\frac{(\beta\gamma x)}{(\alpha\beta\gamma)}, \frac{(\gamma\alpha x)}{(\alpha\beta\gamma)}, \frac{(\alpha\beta x)}{(\alpha\beta\gamma)}\right] \qquad \ldots \quad (2)$$

Further Fundamental Identities.

§ 6. This leads naturally to *compound products.*

A compound inner or outer product is a product of points and primes derived from given points and primes by an invariant process.

Thus, for example, if α, β are two points, any point γ in line with them has relevant co-ordinates

$$[(\gamma\beta), \ (\alpha\gamma)]. \qquad \ldots \quad (1)$$

But $$(\gamma\beta)(\alpha\delta) + (\alpha\gamma)(\beta\delta) = (\alpha\beta)(\gamma\delta)$$

identically. Hence

$$(\gamma\delta) = \frac{(\gamma\beta)}{(\alpha\beta)}(\alpha\delta) + \frac{(\alpha\gamma)}{(\alpha\beta)}(\beta\delta) \qquad \{\delta\} \qquad \ldots \quad (2)$$

$$= \rho(\alpha\delta) + \sigma(\beta\delta) \qquad \{\delta\} \qquad \ldots \quad (3)$$

if $[\rho, \sigma]$ are absolute co-ordinates of γ for the figure $\alpha\beta$.

Now if this last result is written

$$\gamma = \rho\alpha + \sigma\beta,$$

we obtain a statement made familiar sooner or later in any study of vectors. It gives the point γ, on line $\alpha\beta$, linearly in terms of α and β.

This leads to a definition. The outer product $(\gamma\delta\varepsilon \ldots)$ is defined as

$$(\rho\alpha + \sigma\beta.\delta\varepsilon \ldots) = \rho(\alpha\delta\varepsilon \ldots) + \sigma(\beta\delta\varepsilon \ldots). \qquad \ldots \quad VI_1$$

On the left $\rho\alpha + \sigma\beta$ is one of n elements which may be arranged in any order, with proper sign $\pm$. Likewise

$$(\rho\alpha + \sigma\beta.a) = \rho(\alpha a) + \sigma(\beta a). \qquad \ldots \quad VI_2$$

The extension here is obvious. Thus

$$x = \pi\alpha + \rho\beta + \sigma\gamma + \tau\delta$$

arises from the quaternary case when x is a point in the space S_3 defined by a tetrahedron $\alpha\beta\gamma\delta$.

Here

$$\left[\pi, \rho, \sigma, \tau\right] = \left[\frac{(x\beta\gamma\delta)}{(\alpha\beta\gamma\delta)}, \frac{(\alpha x\gamma\delta)}{(\alpha\beta\gamma\delta)}, \frac{(\alpha\beta x\delta)}{(\alpha\beta\gamma\delta)}, \frac{(\alpha\beta\gamma x)}{(\alpha\beta\gamma\delta)}\right]. \quad (5)$$

which give the absolute co-ordinates of x for tetrahedron $\alpha\beta\gamma\delta$.

Similar remarks apply to the correlative case.

Applications to Linear Theorems.

§ 7. (i) Desargue's Theorem. *If two co planar triangles are in perspective their corresponding pairs of sides intersect in three collinear. points ; and conversely.*

If the vertices of the triangles are α, β, γ, and x, y, z respectively, the lines αx, βy, γz meet in a point if

$$(\alpha x \,.\, \beta y \,.\, \gamma z) = 0. \qquad \ldots \quad (1)$$

This is the same as

$$(\alpha\beta y)(x\gamma z) - (x\beta y)(\alpha\gamma z) = 0 \qquad \ldots \quad (2)$$

as is seen by substituting lines

$$\beta y, \;\gamma z, \;\alpha x$$

for lines a, b, $\alpha\beta$ respectively in III_8. Either (1) or (2) is the condition of perspective.

Now the corresponding sides $\beta\gamma$ and yz meet in a point which we can denote by

$$(\beta\gamma y)\, z - (\beta\gamma z)\, y$$

or its equivalent (*cf.* II_6)

$$(yz\beta)\, \gamma - (yz\gamma)\, \beta$$

This point in fact has the equation

$$(\beta\gamma y)(zu) - (\beta\gamma z)(yu) = 0,$$

if the line u passes through it.

The three points

$$\delta = (\beta\gamma y)\, z - (\beta\gamma z)\, y,$$

$$\varepsilon = (\gamma\alpha z)\, x - (\gamma\alpha x)\, z,$$

and $$\zeta = (\alpha\beta x)\, y - (\alpha\beta y)\, x,$$

are in a line if $(\delta\varepsilon\zeta) = 0$, that is (*cf.* VI_1 and VI_2)

$$(\beta\gamma y)(\gamma\alpha z)(\alpha\beta x)(zxy) - (\beta\gamma z)(\gamma\alpha x)(\alpha\beta y)(yzx) = 0 \ldots \quad (3)$$

The full progressive product of these $\delta, \varepsilon, \zeta$ involves eight terms, six of which vanish like one with (zzx) as factor.

Now write βx for a in II_6, and y for δ. Dropping the zero term containing $(\beta\beta x)$ this gives

$$(\beta\gamma y)(\alpha\beta x) = (\alpha\beta\gamma)(\beta x y) + (\beta\alpha y)(\gamma\beta x);$$

Similarly $$(\beta\gamma z)(\gamma\alpha x) = (\alpha\gamma z)(\gamma\beta x) + (x\gamma z)(\gamma\alpha\beta).$$

Hence

$$(xyz)\,[(\alpha\beta\gamma)(\gamma\alpha z)(\beta x y) + (\beta\alpha y)(\gamma\alpha z)(\gamma\beta x)$$

$$- (\alpha\gamma z)(\gamma\beta x)(\alpha\beta y) - (x\gamma z)(\gamma\alpha\beta)(\alpha\beta y)] = 0:$$

or $$(xyz)(\alpha\beta\gamma)(\alpha x\,.\,\beta y\,.\,\gamma z) = 0. \quad \ldots \quad (4)$$

This process is reversible. Accordingly the theorem is true provided $(xyz) \neq 0, \neq (\alpha\beta\gamma)$. The triangles, in fact must not degenerate.

A correlative proof, in terms of sides a, b, c, u, v, w say, runs exactly parallel with this.

(ii) Next suppose for two such triangles the condition

$$(\beta x\,.\,\gamma y\,.\,\alpha z) - (\gamma x\,.\,\alpha y\,.\,\beta z) = 0 \quad \ldots \quad (5)$$

to hold. It can be shewn by exactly similar reasoning that

$$(bu\,.\,cv\,.\,aw) - (cu\,.\,av\,.\,bw) = 0 \quad \ldots \quad (6)$$

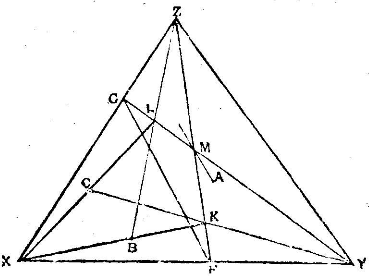
follows, and conversely,

Here $a = \beta\gamma,\ \ b = \gamma\alpha,\ \ c = \alpha\beta,\ \ u = yz,\ \ v = zx,\ \ w = xy.$

In fact (6) expressed in terms of α, β, γ, x, y, z, yields condition (5) with extraneous factors $(\alpha\beta\gamma)$ (xyz).

This leads to a curious theorem on two co-planar triangles ABC and XYZ, namely, that the following property is reciprocal, holding when sides a, b, c and u, v, w, are substituted for vertices A,B,C XYZ :—

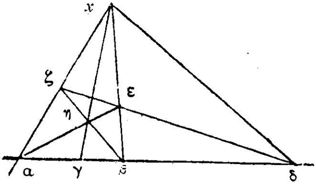
Let BCXYZ be five co-planar points. Then if XB cut YC at K and XC cut ZB at L, and if XY cut ZK at F, and XZ cut YL at G,



then A lies on the line joining the intersection of YG, ZF with that of YZ, GF.

Harmonic Properties of four points in line.

§ 8. Let u be a line through point δ on another line $\alpha\beta$. Let x be a co-planar point not on either line. We construct two points ε, ζ, where



$$\varepsilon = (u\beta)\, x - (ux)\, \beta$$
$$\zeta = (u\alpha)\, x - (ux)\, \alpha$$

and a point γ on $\alpha\beta$ such that $\alpha\varepsilon$, $\beta\zeta$, γx, meet in a point η. Thus

$$\begin{aligned} 0 &= (\alpha\varepsilon \,.\, \beta\zeta \,.\, \gamma x) \\ &= (u\beta)(u\alpha)(\alpha x \,.\, \beta x \,.\, \gamma x) \\ &\quad - (u\beta)(ux)(\alpha x \,.\, \beta\alpha \,.\, \gamma x) \\ &\quad - (ux)(u\alpha)(\alpha\beta \,.\, \beta x \,.\, \gamma x). \end{aligned}$$

This first term vanishes by III_8; and since $(ux) \neq 0$, we get

$$(u\beta)(x\alpha\beta)(\alpha\gamma x) - (u\alpha)(x\alpha\beta)(\beta\gamma x) = 0.$$

Writing $u = \rho\alpha\beta + \sigma\delta x$. ($\rho$, σ numerical) since $(x\alpha\beta) \neq 0$, we finally have

$$(\alpha\gamma x)(\beta\delta x) + (\alpha\delta x)(\beta\gamma x) = 0 \qquad \dots \quad (1)$$

which is the harmonic condition for four collinear points α, β; γ, δ.

If now we refer γ and δ to α and β we can only satisfy (1) by having

$$\gamma = \lambda\alpha + \mu\beta$$
$$\delta = \lambda\alpha - \mu\beta.$$

which gives the usual result in parametric form.

Again in obtaining (1) the point γ can be replaced by any other on the line $x\gamma$, also it is unaltered if we write $\delta + \rho x$ for δ. Hence (1) is the condition that x should subtend a harmonic pencil at four co-planar points α, β, γ, δ.

If we write $\lambda y + \mu z$ for x in (1), (λ, μ numerical) we have a quadratic for $\lambda \,.\, \mu$ in terms of $(\alpha\gamma y)$, $(\beta\delta z)$, ... Thus if y, z are fixed the line yz has two points x satisfying (1). Hence x is a point on a curve of the second degree, *i.e.*, a conic. We merely count the number of x's in a term of (1) to reach this result.

Likewise for higher degrees and for the correlative equation in a, b, c, d, u. For if

$$\alpha x = a, \quad \beta x = b, \quad \gamma x = c, \quad \delta x = d, \quad \alpha\beta = v,$$

the condition (1) becomes

$$(av \,.\, cv \,.\, x)\,(bv \,.\, dv \,.\, x) + (av \,.\, dv \,.\, x)\,(bv \,.\, cv \,.\, x) = 0,$$

i.e., $$(vx)^2\,[(acv)\,(bdv) + (adv)\,(bcv)] = 0,$$

i.e., $$(acv)\,(bdv) + (adv)\,(bcv) = 0. \qquad \ldots \quad (2)$$

In the usual notion of geometry if $(ABCD)$ is a harmonic range then X $(ABCD)$ is a harmonic pencil.

Exactly similar reasoning applies for higher dimensions. Thus in S_3,

$$(acuv)\,(bduv) + (aduv)\,(bcuv) = 0. \qquad \ldots \quad (3)$$

if four planes a, b, c, d, through a line p cut another line $u\,v$ harmonically. And so on.

Three Dimensional Examples.

§ 9. The equation of the plane through three points $\alpha\beta\gamma$ is

$$(\alpha\beta\gamma x) = 0.$$

That of the point on three planes a, b, c is

$$(abcu) = 0. \qquad \ldots \quad (2)$$

The line joining two points xy is denoted by p. If p is common to two planes u, v we write the condition in various ways :—

$$\left.\begin{array}{rlll} & (xy\alpha\beta) = \lambda\,(p\alpha\beta) & \{\alpha, \beta\} \\ & (uvab) = \mu\,(pab) & \{a, b\} \\ (ax)\,(by) - (ay)\,(bx) = & (xy \,.\, ab) = \lambda\,(pab) & \{a, b,\} \\ & (uv \,.\, \alpha\beta) = \mu\,(p\alpha\beta) & \{\alpha, \beta\} \end{array}\right\} \ldots \quad (3)$$

$\lambda, \mu \neq 0$. Very often they may be taken as unity, $\lambda = \mu = 1$.

Shortly, $$xy = \lambda p, \quad uv = \mu p. \qquad \ldots \quad (4)$$

The equation of the line through two points α, β is

$$(\alpha\beta p) = 0, \qquad \ldots \quad (5)$$

p denoting any line which intersects it: correlatively for $(abp) = 0$.

Quadric Surfaces defined by lines and points.

§ **10.** *The point x lies on a quadric surface, containing three lines $\alpha\beta$, $\gamma\delta$, $\varepsilon\zeta$ entirely, if*

$$(x\alpha\beta\gamma)(\delta\varepsilon\zeta x) - (x\alpha\beta\delta)(\gamma\varepsilon\zeta x) = 0. \qquad \ldots (1)$$

For, this condition cannot be deduced from III_C, and therefore does not vanish identically for all positions of x.

But it is satisfied (*cf.* VI_1) by

$$x = \lambda\alpha + \mu\beta,\ \lambda\gamma + \mu\delta,\ \lambda\varepsilon + \mu\zeta \qquad :,\ \mu$$

and so the surface, which is of second degree, contains these three lines.

It is interesting to develop the theory of generating lines from this starting point.

Next, if $\alpha\beta = p$, $\gamma\delta = q$, $\varepsilon\zeta = r$.

we write this condition (1) as

$$(xpqrx) = 0 \qquad \ldots (2)$$

which defines a new type of magnitude $(xpqrx)$ alternating (*cf.* II_1) in p, q, r, as is easy to verify. It is however only $(ab\,.\,\alpha\beta)$ of II_0, in disguise. In fact let plane a be xq and b be xr. Then

$$xq = x\gamma\delta.$$

So
$$\begin{aligned}(ab\,.\,\alpha\beta) &= (xqxr\,.\,\alpha\beta), \text{ say}\\ &= (xq\alpha)(xr\beta) - (xq\beta)(xr\alpha)\\ &= -(\overline{xq}prx)\\ &= (xpqrx). \qquad \ldots (3)\end{aligned}$$

Correlatively for $(upqru) = 0. \qquad \ldots (4)$

§ 11. Next if we write

$$\begin{aligned}U &= (x\ \ \alpha\beta\,.\,\gamma\delta\,.\zeta x)\\ V &= (x\alpha\delta\,.\,\beta\varepsilon\,.\,\gamma\zeta x)\\ W &= (x\delta\varepsilon\,.\,\zeta\alpha\,.\,\beta\gamma x)\end{aligned}$$

And expand as above into two terms each, we find the six terms of $U + V - W$ cancel in pairs.

Hence

$$U + V - W = 0,$$

shewing that three such quadrics have a common curve of intersection. Thus

"*If $\alpha\beta\gamma\delta\epsilon\zeta$ are six points of a skew hexagon the three quadrics containing the two sets of alternate sides and the set of diagonals respectively have a common curve of intersection.*"*

Dropping the x in the expressions for U, V, W, we obtain a *ternary* identity

$$U_0 + V_0 - W_0 = 0.$$

where $U_0 = (\alpha\beta\,.\,\gamma\delta\,.\,\epsilon\zeta)$

$$= (\alpha\beta\gamma)\,(\delta\epsilon\zeta) - (\alpha\beta\delta\ (\gamma\epsilon\zeta),$$

etc. This U_0 vanishes if the six points are co-planar and the lines $\alpha\beta$, $\gamma\delta$, $\epsilon\zeta$, meet in a point. This practically leads to Pappus Theorem, (a special case of Brianchon's Theorem for a conic),

If $\alpha\beta$, $\gamma\delta$, $\zeta\epsilon$ are concurrent, and $\alpha\delta$, $\beta\epsilon$, $\gamma\zeta$ are, so also are $\beta\gamma$, $\delta\epsilon$, $\zeta\alpha$.

The dropping of the x from the expression U, V, W is the simple symbolic equivalent of projection from a point x, common to all three quadrics, on to a plane not containing x.

Three Quadrics U, V, W in general.

§ **12.** Such surfaces have eight points in common, any one of which is uniquely determined by the other seven. Consider the determinant

$$X = \begin{vmatrix} (1256)\ (3476), & (1258)\ (3478) \\ (1376)\ (3456), & (1278)\ (3458) \end{vmatrix} \qquad \dots \quad (1)$$

where the eight digits denote the points. We write this shortly as

$$X = (12\,.\,34\,.\,5678)^2 \qquad \dots \quad (2)$$

If α denote the point where line 56 meets the plane 348, let us write

$$\alpha = \{\,348,\ 56\,\},\ \text{also}\ \beta = \{\,812,\ 67\,\}.$$

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* Cf. Baker: *Journ. London Mathc. Soc.* 1. (1926) p. 181.

Then a short calculation* shows

$$X \equiv (\alpha\beta \, . \, 125 \, . \, 734) \quad \dots \quad (3)$$

So that X vanishes if the common line of planes 125, 734 meets the line $\alpha\beta$.

This function X is of considerable algebraic interest.

If any pair of digits 5, 6, 7, 8 are interchanged, X changes sign. So also if (12) are interchanged simultaneously with (34).

X is symmetrical in 1 and 2, and also in 3 and 4. Three such functions properly chosen satisfy the identity.

$$(12 \, . \, 34 \, . \, 5678)^2 + (12 \, . \, 35 \, . \, 4678)^2 + (12 \, . \, 45 \, . \, 3678)^2 = 0. \quad \dots \quad (4)$$

If 8 is replaced by x, then $X = 0$ *is the equation of a quadric surface through the other seven points, and containing two lines* 12, 34. This is easy to verify directly by writing $x = \lambda 1 + \mu 2, \lambda 3 + \mu 4$, 5, 6, 7 in turn.

Special Case. The Weddle Surface.

§ 13. Let points 2, 3 coincide at a point called 0. Then (1) becomes

$$(0156)(0476)(0178)(0458) - (0158)(0478)(0176)(0456).$$

This vanishes if point 0 is the vertex of a quadric cone containing the other six points. For if we project these six points from 0 on to an external plane we get

$$(1'5'6')(4'7'6')(1'7'8')(4'5'8') - (1'5'8')(4'7'8')(1'7'6')(4'5'6') = 0,$$

for the property of the six co-planar projected points.

This holds if points $\{1'5', 6'4'\}$, $\{1'7', 8'6'\}$, $\{7'4', 5'8'\}$ are in line, as is proved by use of conditions such as II_6.

This gives *Pascal's Theorem* for a hexagon inscribed in a conic, which also proves the cone with vertex 0 to be a quadric. Since 0 occurs four times in each term of X the locus of 0 is a quartic Weddle surface; in fact it goes by the name of its discoverer.

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* Cf. *Proc. Cambridge Phil. Soc.* XXII (1925). pp 481—487.

§ **14.** Returning to the general form $X = (12.34.5678)^2$ consider $X = 0$ as a locus of the line 12, called p, with points 3, 4, 5, 6, 7, 8 fixed. Here is an example of a *quadratic line complex.* We have just verified that all such lines through point 3 lie on a quadric cone containing points 3, 4, 5, 6, 7, 8. Such is called the complex cone of the point 3 for the quadratic complex.

Similarly the cone of the point 4 has 4 for vertex and contains also 3, 5, 6, 7, 8. So these cones have 34 as common line and therefore also a twisted cubic curve of intersection.

This leads to a (2, 2) congruence denoted by conditions

$$(p.34.5678)^2 = (p.35.4678)^2 = (p.45.3678)^2 = 0 \quad \ldots \quad (1)$$

which are in effect two independent equations owing to an identity like § 12 (4). *This congruence is satisfied by all chords of the twisted cubic through the points 3, 4, 5, 6, 7, 8 together with all lines through one or other of the points 6, 7, 8.* For the line 12 satisfies these equations if 2 coincides with 6 or 7 or 8, wherever 1 may be. Also since $(p\ 34) \neq 0$ in general, the quadric $X = 0$ has lines p and 34 as generators of one system. The three quadrics (1) have p in common and therefore a twisted cubic also. This cubic cuts line 34 twice, at 3 and at 4 in fact; hence it also cuts p twice. Hence p is its chord.

This also proves that *a twisted cubic through six of eight associated points (common to three arbitrary quadrics) has the line joining the remaining two points as chord.*

Quadric Surface defined by seven points and one line.

§ 15. The condition that seven points 1 2 3 4 5 6 7 and a line p should lie on a quadric is

$$\begin{aligned} G = {} & (1234)\,(1567)\,(26\,p)\,(73\,p)\,(45\,p) \\ & + (1245)\,(1673)\,(27\,p)\,(34\,p)\,(56\,p) \\ & + (1256)\,(1734)\,(23\,p)\,(45\,p)\,(67\,p) \\ & + (1267)\,(1345)\,(24\,p)\,(56\,p)\,(73\,p) \\ & + (1273)\,(1456)\,(25\,p)\,(67\,p)\,(34\,p) = 0 \quad \ldots \quad (1) \end{aligned}$$

For the expression G does not vanish identically, and regarded as a locus of the point 2 it is seen to be satisfied if 2 coincides with 1, 3, 4, 5, 6, 7 or any point on line p. Also if the seven points are given, $G = 0$ *is the cubic complex of lines forming all generators of the* ∞^2 *quadric surfaces.*

$$\lambda U + \mu V + \gamma W$$

passing through the seven points.

If now the point 1 occupy position

$$\xi 2 + \eta 3 + \zeta 4 \qquad \ldots \quad (2)$$

in the plane 234, where ξ, η, ζ are numerical, the expression G simplifies to

$$A \eta\zeta + B \zeta\xi + C \xi\eta \qquad \ldots \quad (3)$$

where it is found that

$$\begin{aligned} A &= -(34p)\{34.p.2567\}^2 \\ B &= -(42p)\{42.p.3567\}^2 \\ C &= -(23p)\{23.p.4567\}^2 \end{aligned} \qquad \ldots \quad (4)$$

using the notation of § 12. Here we have the *associated functions*, like $\{34.p.2567\}^2$, occurring again. Owing to identity § 12 (4) we now write

$$\frac{A}{(34p)} + \frac{B}{(42p)} + \frac{C}{(23p)} = 0. \qquad \ldots \quad (5)$$

In this we have an example of relevant co-ordinates (§ 5), for if 234 is a triangle of reference, the expression G as simplified in (3) leads to the following equation for the conic section of our quadric surface by plane 234:—

$$(PP'q)\, \xi'\eta\zeta + (QQ'q)\, \eta'\zeta\xi + (RR'q)\, \zeta'\xi\eta = 0 \qquad \ldots \quad (6)$$

where P, Q, R, P′, Q′, R′ are the points

$$\begin{array}{lll} P = \{65, 347\} & Q = \{65, 427\} & R = \{65, 237\} \\ P' = \{25, p7\} & Q' = \{35, p7\} & R' = \{45, p7\} \end{array}$$

and q is the line $\{234, 6p\}$, while (ξ', η', ζ') are co-ordinates of the point $\{234, p\}$. The expression $(PP'q)$ is in fact $-A/(34p)$.

From this equation it is easy to deduce a ruler construction for points on the conic. A little further enquiry would show that it also leads to a ruler construction for any number of points on a quadric surface through six given points and a given line.

The corresponding expression for the condition that ten points should lie on a quadric is

$$\Sigma\ (1234)\,(1567)\,(2689)\,(7390)\,(4508) = 0$$

where Σ denotes a sum of 240 such terms whose law of formation has been determined*.

Conditions for four lines to lie on a quadric surface.

§ 16. Let p, q, r, s be four non-intersecting lines. Then p lies on the surface $(x\ q\ r\ s\ x) = 0$ containing lines q, r, s if $x = up\ \{u\}$. For in this case x can be any point of line p. Hence

$$(u\ p\,.\,q\ r\ s\,.\,up) \equiv 0 \qquad \{u\}. \qquad \ldots\ (1)$$

Similarly q lies on surface $(x\ r\ s\ p\ x) = 0$, if

$$(u\ q\,.\,r\ s\ p\,.\,u\ q) \equiv 0 \qquad \{u\} \qquad \ldots\ (2)$$

A straight forward use of the fundamental identities of § 3 and § 10, shows that in these expressions any *adjacent* pair of letters p, q, r, s may be interchanged by the following rule

$$(\theta\ r\ s\ i) + (\theta\ s\ r\ i) + (r\ s)\,(\theta\ i) = 0 \qquad \ldots\ \text{XVI}_1$$

where, *e.g.* θ denotes $u\ p\ q$ and i denotes $u\,p$ in (1).

Shifting p through from left to right in (1) we obtain

$$(pq)\,(u\ r\ s\ p\ u) + (p\ r)\,(u\ q\ p\ s\ u) + (p\ s)\,(u\ p\ q\ r\ u) = 0 \ \ldots\ (3)$$

since $(pp) = 0$. If now we write

$$P = (u\ q\ r\ s\ u) \qquad R = (u\ p\ q\ s\ u)$$
$$Q = (u\ p\ s\ r\ u) \qquad S = (u\ q\ p\ r\ u)$$

* A full account of these problems is given in a thesis by Dr. Alfred Young and the author. *Trans. Cambridge Phil. Soc.* 23 (1926) pp. 265—301.

we have

$$\left.\begin{array}{l}(p\,q)\,Q + (p\,r)\,R + (p\,s)\,S = 0\\(p\,q)\,P + (q\,r)\,R + (q\,s)\,S = 0\\(p\,r)\,P + (q\,r)\,Q + (r\,s)\,S = 0\\(p\,s)\,P + (q\,s)\,Q + (r\,s)\,R = 0\end{array}\right\} \qquad \ldots \quad (4)$$

whence by elimination

$$\sqrt{(p\,q)\,(r\,s)} + \sqrt{(p\,r)\,(q\,s)} + \sqrt{(p\,s)\,(q\,r)} = 0. \qquad \ldots \quad (5)$$

If *one* of conditions (4) holds for all positions of plane u, the four lines lie on a quadric. If *two* conditions (4) hold, they lie in one generating system.

Condition (5) *is found to be satisfied by all tangent lines* p *to the quadric* $(x\,q\,r\,s\,x) = 0$. Each factor $(p\,q)$ is the mutual moment of two lines.

A question may be asked: what does

$$\{(p\,q)\,(r\,s)\}^{\frac{1}{n}} + \{(p\,r)\,(q\,s)\}^{\frac{1}{n}} + \{(p\,s)\,(q\,r)\}^{\frac{1}{n}} = 0 \text{ imply?}$$

Commentary.

§ **17.** In the above examples some things have been merely sketched without full explanation. But the reader possesses in the fundamental definitions and identities (indicated by Roman numerals for reference) all the necessary appliances for the proof. The magnitudes in brackets, $(\alpha\beta)$, $(\alpha\beta\gamma)$, etc. arise naturally in the symbolic invariant theory, and in developing some years ago a purely algebraic notation for quaternary forms, I found later that I was in fact constructing symbols which were used by Grassman in the *Ausdehnungslehre* (1841), (1862).*

It is perhaps well to say that the magnitude $(\alpha\beta)$ does not mean the metrical distance between points α and β. It generalises on, and includes as special interpretations, metrical notions. Except for the introduction of such magnitudes the treatment is entirely projective. But an adequate discussion of this would carry one too far afield and cannot be undertaken here.

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* *Cf.* Whitehead: *Universal Algebra* (Cambridge, 1895).

SOME CONFIGURATIONS OF POINTS ON A CIRCLE*

BY T. VIJAYARAGHAVAN.

1. The object of this note is to discuss the existence of certain configurations of points on the circumference of a circle, which involve in their definitions partly combinatory moments and partly such connected with the theory of sets of points. The interest of this discussion lies perhaps primarily in its connection with some theorems on the existence of certain configurations of singular points on the circle of convergence. We must therefore, before formulating the results, state these connections.

Mr. Mandelbrojt † has recently proved that if in a power series

$$f(z) = \Sigma a_n z^n$$

with a finite radius of convergence, all co-efficients corresponding to a certain class of residues modulo p vanish, then the function has on the circle of convergence at least two singular points, and further that to every singular point α on that circle there exists at least one other singular point β such that their ratio is a pth root of unity. His proof proceeds by assuming Hadamard's well-known theorem ‡ on the multiplication of singularities. Dr. Ostrowski § has indicated that the theorem follows at once from the following simple identity:—

$$(1) \qquad \sum_{r=0}^{p-1} \varepsilon^{-ar} f(\varepsilon^r z) = p \sum_{\substack{n \equiv a \\ (\text{mod } p)}} a_n z^n$$

where ε is a primitive pth root of unity.

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* I am much indebted to Dr. Ostrowski for suggesting to me the problem of this note and pointing out its connection with other results.

† Mandelbrojt: *Acta Math.* 45 (1925) 129—142.

‡ For a proof, see J. Hadamard: Un theoreme sur les series entieres *Acta Mathematica* 22 (1898).

§ Ostrowski: *Mathematische Miscellen* VII. Uber singularitaten gewisser mit Lucken behafteten Potenzreihen; *Jahresber d. Deutchen Math. Ver* 3.

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2. More generally if we assume that an analogous property holds for a number of mutually prime numbers $p_1, p_2, \dots p_r$, then there exists a corresponding result stated completely by Dr. Ostrowski, *viz.*

"*To every singular point* α_1 *on the circle of convergence there correspond* $2^r - 1$ *distinct singular points which have also the property, that the quotients by* α_1 *are roots of unity*"* Further he indicates a structural property of such a configuration which may be thus described.

The function has besides α_1, the distinct singular points $\alpha_2\ \alpha_3 \dots \alpha_{2^r}$ such that

$$(\alpha_1/\alpha_2)^{p_1} = 1,$$

$$(\alpha_1/\alpha_3)^{p_2} = 1,\ (\alpha_2/\alpha_4)^{p_2} = 1,$$

$$(\alpha_1/\alpha_5)^{p_3} = 1,\ (\alpha_2/\alpha_6)^{p_3} = 1,\ \dots\ (\alpha_4/\alpha_8)^{p_3} = 1,$$

$$\dots \qquad \dots \qquad \dots \qquad \dots$$

$$\dots \qquad \dots \qquad \dots \qquad \dots$$

$$(\alpha_1/\alpha_{2^{r-1}+1})^{p_r} = 1,\ \dots\ \dots (\alpha_{2^{r-1}}/\alpha_{2^r})^{p_r} = 1.$$

We shall refer to such a configuration as an *ordered* configuration.

The order of the numbers $p_1, p_2, \dots p_r$ can here be prescribed arbitrarily. It was also shown by an example that the above result is the best of its kind. Consider the function

$$f(z) = \frac{1}{1-z} - \sum_{n \leq r} \frac{1}{1-e_n z} + \sum_{n_1 < n_2 \leq r} \frac{1}{1-e_{n_1} e_{n_2} z}$$

$$- \sum_{n_1 < n_2 < n_3 \leq r} \frac{1}{1-e_{n_1} e_{n_2} e_{n_3} z} + \dots \dots \dots$$

where $e_n = e^{2\pi i/p_n}$. This function has precisely 2^r poles, and the power series that corresponds to it is such that the co-efficients a_n vanish whenever n is a multiple of one or more of the numbers $p_1, p_2, \dots p_r$.

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* Mandelbrojt has already pointed out that under these conditions there are at least $r+1$ singular points. *Acta Math.* 45 (1925), 129—143.

3. Similar results are also proved even if there be an infinite number of non-vanishing a_n from these classes provided that a certain condition is satisfied, namely, if $k/n_k \rightarrow 0$ when $k \rightarrow \infty$, where

$$a_{n_1}, \ a_{n_2} \ \dots \ a_{n_k}$$

are all the numbers of the classes that do not vanish*. It has been proved under these conditions that there is at least one *ordered* configuration of 2^r singular points †.

In this connection Mr. Hardy has pointed out that a direct application of the identity (1) leads to the general conclusion :

"*If for an integer* p *the aggregate of the terms of* f (z) *from a class of residues to modulus* p *has the circle of convergence of* f (z) *as a natural boundary, and if* ε *is a primitive* p*th root of unity, then to every* α *on the circumference of the circle of convergence, the corresponding cycle,* α, $\varepsilon\alpha$, ... $\varepsilon^{p-1}\alpha$ *contains at least one singular point of* f (z)"

4. This result states the existence of an *infinite* number of singularities, but does not indicate the existence of an *ordered* configuration of singular points. However in the case $r = 1$, Dr. Ostrowski has shown ‡ that a set of singular points on the circle of convergence obtained from this result contains at least one *ordered* configuration. The object of this note is to investigate whether we can deduce similar configurations for $r \geqslant 2$. It is shown here that also when $r = 2$, the set obtained from Hardy's result contains an *ordered* configuration, but that

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* In this connection reference may be made to a paper by F. Carlson and E. Landau published in *Gottinger Nachrichten* (1921) heft 2, (184—188) which contains a simple proof of the theorem of Fabry that

"*If* $k/n_k \rightarrow 0$, *and* $\sum a_{n_k} z^{n_k}$ *has a finite radius of convergence, then the circumference of that circle is a natural boundary for the function defined by the series.*"

† Mandelbrodjt stated without proof [C. R. (1924) 636], that if an $\varepsilon > 0$ exists such that $\sum 1/n_k^{1-\varepsilon}$ converges, then to *every* singular point α there exists another singular point β such that their ratio is a *p*th root of unity. But this does not follow from the proof given by Ostrowski for his theorem.

‡ *loc. cit.*

a similar result is not always true when $r>2$. Theorem 1 of this note discusses the case $r=2$, and proves the existence of an *ordered* configuration. It is also pointed out by an example that the order of p_1 and p_2 cannot always be taken arbitrarily in deriving the *ordered* configuration. Next, by another example, it is shown that Hardy's result need not lead to an *ordered* configuration when $r \geqslant 3$. It is however shown that when $r=3$, we can derive an *ordered* configuration by adding to Hardy's set, at the most, two other points.

5. Before proceeding to Theorem 1, I shall state how Dr. Ostrowski derives his results mentioned in § 3, since this argument has not been included in his paper cited above. The argument depends upon the following well-known theorem of G. Faber.*

If n_k be a sequence of positive integers such that $k/n_k \to 0$ as $k \to \infty$, then the integral function

$$\phi(z) = \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{n_k^2}\right)$$

possesses the following properties:—

(A) *If m_k run through all positive integers different from n_k, then*

$$\sqrt[m_k]{\phi(m_k)} \to 1.$$

(B) If $f(z) = \Sigma a_n z^n$ *be a power series with radius of convergence equal to unity, then every singular point of the power series $\Sigma a_n \phi(n) z^n$ on the unit circle is also a singular point of $f(z)$.*

Now let us consider the numbers n_k mentioned in § 3, and the corresponding functions $\phi(z)$ and $\Sigma a_n \phi(n) z^n$. If the latter power series has the radius of convergence unity, then it has, as mentioned in § 2, an ordered configuration of singularities and the same holds therefore for $f(z)$. If the radius of convergence of $\Sigma a_n \phi(n) z^n$ is greater than unity, then obviously the original series is a sum of a power series with radius of convergence greater than unity, and of $\Sigma a_{n_k} z^{n_k}$. The radius of convergence of $\Sigma a_n z^n$ is unity, and therefore that of $\Sigma a_{n_k} z^{n_k}$ is also unity.

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* G. Faber: *Sitzungsber. d. Munch Akad. der Wissenschaften*, 36 (1906).

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NOTES AND QUESTIONS.

Notes and Questions.

Poristic Relations between two Circles in the Non-Euclidean Plane.

It is well-known that the radii and the distance between the centres of two circles have to satisfy the relation

$$d^2 = R^2 \pm 2Rr,$$

in order that there may exist triangles which are inscribed in one and circumscribed to the other. An analogous relation has also to be satisfied in the case of two small circles on the surface of a sphere. It is proposed here to derive this relation by invariant methods in the Non-Euclidean plane, in which the metric is determined by an absolute conic Ω, and circles are conics having double contact with Ω.

1. Let S_1, S_2 be two circles of radii r, R, and Ω the Absolute. We may take the equation of Ω in the form :

$$\Omega: \qquad x^2 + y^2 + k^2z^2 = 0,$$

where k is the absolute linear constant of the space. If the centres of S_1, S_2 are at $(x_1y_1z_1)$, $(x_2y_2z_2)$, their equations are then of the form :

$$S_1: (x_1^2+y_1^2+k^2z_1^2)(x^2+y^2+k^2z^2)\cos^2\frac{r}{k} - (xx_1+yy_1+k^2zz_1)^2 = 0$$

$$S_2: (x_2^2+y_2^2+k^2z_2^2)(x^2+y^2+k^2z^2)\cos^2\frac{R}{k} - (xx_2+yy_2+k^2zz_2)^2 = 0.*$$

The invariant condition, that there may exist a triangle circumscribed to S_1 and inscribed in S_2, is, in the usual notation

$$\theta_1^2 = 4\,\Delta_1\,\theta_2.$$

We find

$$\Delta_1 = -k^2\,\omega_{11}^3\cos^4\frac{r}{k}\sin^2\frac{r}{k}$$

$$\theta_1 = k^2\,\omega_{11}\cos^2\frac{r}{k}\left[\omega_{11}\,\omega_{22}\left(\alpha - \cos^2\frac{R}{k}\right) - \omega^2_{12}\right]$$

$$\theta_2 = k^2\,\omega_{22}\cos^2\frac{R}{k}\left[\omega_{11}\,\omega_{22}\left(\alpha - \cos^2\frac{r}{k}\right) - \omega^2_{12}\right].$$

* Sommerville : *Non-Euclidean Geometry*, page 136.

where

$$\omega_{11} = x_1^2 + y_1^2 + k^2 z_1^2$$

$$\omega_{22} = x_2^2 + y_2^2 + k^2 z_2^2$$

$$\omega_{12} = x_1 x_2 + y_1 y_2 + k^2 z_1 z_2$$

$$a = 1 - \cos^2\frac{r}{k} - \cos^2\frac{R}{k} + 3\cos^2\frac{r}{k}\cos^2\frac{R}{k}.$$

Substituting in the invariant relation, and using the fact that

$$\omega_{11}\ \omega_{22}\cos^2\frac{d}{k} = \omega^2_{12},\ *$$

where d is the distance between the centres, we obtain the relation :

$$\left[\sin^2\frac{d}{k} - \left(\cos^2\frac{r}{k} + 2\cos^2\frac{R}{k} - 3\cos^2\frac{r}{k}\cos^2\frac{R}{k}\right)\right]^2$$

$$+ 4\sin^2\frac{r}{k}\cos^2\frac{R}{k}\left[\sin^2\frac{d}{k} - \left(\cos^2\frac{R}{k} + 2\cos^2\frac{r}{k} - 3\cos^2\frac{r}{k}\cos^2\frac{R}{k}\right)\right] = 0.$$

This easily works out to :

$$\sin^2\frac{d}{k} = \sin^2\frac{R}{k}\cos^2\frac{r}{k} \pm \tfrac{1}{2}\sin\frac{2r}{k}\sin\frac{2R}{k}. \qquad \text{... (B)}$$

If k is real (so that the Absolute is imaginary, and the space, elliptic), then this gives the sought real relation between the circles ; if the space is hyperbolic, then k is purely imaginary and equal to ik', and in the above relation k will have to be replaced by k', and the circular functions by the corresponding hyperbolic functions.

It will be noticed that when $k \to \infty$, the relations (B) becomes the usual Euclidean relation

$$d^2 = R^2 \pm 2Rr.$$

We may easily transform (B) into the following shape, given by Casey :†

$$\cos^2\frac{d}{k} - \cos^2\frac{R}{k} = \cos^2\frac{R \pm r}{k} - \cos^2\frac{R}{k}\cos^2\frac{r}{k}.$$

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* Sommerville, page 129.

† Casey : *Spherical Trigonometry*, p. 80, Questions 18 & 19.

The form of (B) for hyperbolic space, is

$$\sinh^2 \frac{d}{k} = \sinh^2 \frac{R}{k} \cosh^2 \frac{r}{k} \mp \frac{1}{2} \sinh \frac{2r}{k} \sinh \frac{2R}{k}. \qquad \ldots \quad (C)$$

We may use this relation to find the inradius r of a triangle of maximum area in hyperbolic space. It is well-known that every two of the sides of a triangle of maximum area are parallel, so that its circumcircle coincides with the Absolute, and its circumcentre is indeterminate. Thus R, and therefore $\sinh \frac{R}{k}$, $\cosh \frac{R}{k}$ become infinite, while d may be supposed to be an arbitrary finite quantity. In order that d may be finite in :

$$\sinh^2 \frac{d}{k} = \sinh^2 \frac{R}{k} \left[\cosh^2 \frac{r}{k} \pm 2 \cosh \frac{r}{k} \sinh \frac{r}{k} \right],$$

we must have

$$\cosh \frac{r}{k} \pm 2 \sinh \frac{r}{k} = 0,$$

or

$$r = k \tanh^{-1} (\pm \tfrac{1}{2}) = \tfrac{1}{2} k \log_e 3 \,*.$$

2. We shall now obtain the form which (C) takes in the interesting particular cases when S_1 or S_2 is a horocycle.

Case i. *S_1 becomes a horocycle.*

In this case R remains finite. but d and r must tend to infinity in such a way that

$$d - r = \text{a finite quantity } \lambda.$$

We therefore substitute $r + \lambda$ for d in (C), and let $r \to \infty$. We choose the ambiguous sign in (C) as $+$, since the horocycle S_1, must be an *ex-circle*, of each of the in-circum-triangles. We thus obtain the relation;

$$e^{\frac{2\lambda}{k}} = \sinh \frac{R}{k} \left\{ \sinh \frac{R}{k} + 2 \cosh \frac{R}{k} \right\} \qquad \ldots \quad (E)$$

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* Sommerville : *Ibid.* Ex. II, Question 10.

Case ii. S_2 *becomes a horocycle.*

Here r is finite, and d and R become infinite in such a way that $d - R$ is a finite quantity λ. Substituting $R + \lambda$ for d, and making R infinite, we obtain a similar relation :

$$e^{\frac{2\lambda}{k}} = \cosh \frac{r}{k} \left\{ \cosh \frac{r}{k} \pm 2 \sinh \frac{r}{k} \right\} \qquad \dots \quad (F)$$

Case iii. *Both S_1 and S_2 become horocycles.*

For this case, we start with (E), and consider that both λ and R become infinite in such a way that $\lambda = R + \mu$, where μ is finite The geometrical meaning is clearly that μ is the length of the intercept made by the horocycles S_1, S_2, on the unique line whose two directions are parallel to their respective axes. Substituting $R + \mu$ for λ and making R infinite in (E), we obtain the relation :

$$e^{\frac{2\mu}{k}} = \tfrac{3}{4}, \text{ or } \mu = \tfrac{1}{2} k \log_e \tfrac{3}{4} \qquad \dots \quad (G)$$

The negative sign of μ expresses, that the two infinite radial segments on the common diameter of the horocycles, separate one another.

G. V. KRISHNASWAMY AIYANGAR.

K. SREENIVASARAGHAVAN.

Gauss' Formula and allied results.

1. (The results given in paragraphs (2) and (4) have been long known. They are due to Dedekind and Liouville. Several proofs have been given, of which perhaps the most direct is that given in § 152 (pp. 577, et seq.) of Vol II of Landau's "*Handbuch der Lehre von der Verteilung der Primzahlen.*"

The method of proof, which is based on the multiplication of Dirichlet's Series, is, I believe, new. The following are further references :—

(1) *Encyclopédie des Sciences Mathématiques.* Tome I. (troisieme volume) *Theorie des Nombres,* pp. 232 and 268 where further references are given.

(2) *Philosophical Magazine*, Vol. 18 (1884). Fifth Series, pp. 518—540.

(3) *Journal of the Indian Mathematical Society*, Vol. X. Prof. Hallberg : *Arithmetical Inversion.*)

2. Notation.

$\lambda(n)$ denotes the number of divisors of n;

$\sigma_a(n)$ is the sum of the a'th powers of the divisors of n;

$\mu(n) = 0$ if n has any repeated prime factor, and is $+1$ or -1 according as n has an even or odd number of distinct prime factors;

$\phi(n)$ is the number of integers less than n and prime to it.

Let $f(x)$ be a suitable function of x. Also let $F(N) = \Sigma f(d_r)$, where $d_1, d_2 \ldots$ etc. are divisors of N, and the summation extends over all the divisors.

Now,
$$\sum \frac{f(n)}{n^s} \times \zeta(s) = \sum \frac{F(n)}{n^s}$$

$$\therefore \quad \sum \frac{f(n)}{n^s} = \sum \frac{\mu(n)}{n^s} \times \sum \frac{F(n)}{n^s}$$

Equating co-efficients of n^{-s},

$$\Sigma\, \mu(d_1)\, F(d_2) = f(d_1 d_2) = f(n)$$

where the summation on the left extends to all pairs d_1, d_2 of conjugate divisors of n. ... (2)

Also $\sum \frac{1}{n^s} \times \sum \frac{\mu(n)}{n^s} = 1 \;\therefore \sum \mu(d_r) = 0$ unless n is unity. ... (3)

(Mathew's *Theory of Numbers*, Part I, p. 313).

Also, we know:
$$\frac{\zeta(s-1)}{\zeta(s)} = \sum \frac{\phi(n)}{n^s}$$

$$\therefore \quad \zeta(s-1) = \sum \frac{n}{n^s} = \sum \frac{\phi(n)}{n^s} \times \sum \frac{1}{n^s}$$

Therefore as before, $\sum_{(d)} \phi(d_r) = n$ which is Gauss' formula. ... (4)

3. Some of the results in this paragraph, are, I believe, new.

We have: $\zeta(s)\,\zeta(s-1) = \sum \frac{\sigma_1(n)}{n^s}$ and $\frac{\zeta(s-1)}{\zeta(s)} = \sum \frac{\phi(n)}{n^s}$;

Multiplying these we have

$$\sum \frac{n\,\lambda(n)}{n^s} = \{\zeta(s-1)\}^2 = \sum \frac{\sigma_1(n)}{n^s} \times \sum \frac{\phi(n)}{n^s}$$

$$\therefore \qquad \Sigma\,\phi(d_1)\,\sigma_1(d_2) = n\,\lambda(n) \qquad \ldots \quad (5)$$

e.g if $n = 30$, each side is 240.

On squaring both sides of the result:

$$\zeta(s)\,\zeta(s-a) = \sum \frac{\sigma_a(n)}{n^s},$$

we get
$$\{\zeta(s)\}^2 \times [\zeta(s-a)]^2$$

$$= \sum \frac{\lambda(n)}{n^s} \times \sum \frac{n^a\,\lambda(n)}{n^s} = \left\{\sum \frac{\sigma_a(n)}{n^s}\right\}^2$$

$$\therefore \qquad \Sigma\,\lambda(d_1)\,d_2^a\,\lambda(d_2) = \Sigma\,\sigma_a(d_1)\,\sigma(d_2) \qquad \ldots \quad (6)$$

e.g., if $n=6$, $a=1$, both sides $= 48$; while if $a = 2$, each side is 200.

$$\text{Since } [\zeta(s-1)]^2 \times \frac{1}{\zeta(s)} = \sum \frac{n\,\lambda(n)}{n^s} \times \sum \frac{\mu(n)}{n^s}$$

$$= \frac{\zeta(s-1)}{\zeta(s)} \times \zeta(s-1) = \sum \frac{\phi(n)}{n^s} \times \sum \frac{n}{n^s}$$

we get
$$\Sigma\,\mu(d_1)\,.\,d_2\,\lambda(d_2) = \Sigma\,d_1\,\phi(d_2) \qquad \ldots \quad (7)$$

e.g., for $n = 6$, each side is 15.

Again:
$$\sum \frac{\phi(n)}{n^s} \times \sum \frac{\lambda(n)}{n^s} = \frac{\zeta(s-1)}{\zeta(s)} \times [\zeta(s)]^2$$

$$= \zeta(s-1)\,\zeta(s) = \sum \frac{\sigma_1(n)}{n^s}$$

$$\therefore \qquad \Sigma\,\phi(d_1)\,\lambda(d_2) = \sigma_1(n). \qquad \ldots \quad (8)$$

(*Vide* Professor Hallberg's paper cited above.)

Again $\{\zeta(s)\}^2 \times \zeta(s-a)$

$$= \sum \frac{\sigma_a(n)}{n^s} \times \zeta(s) = \sum \frac{\sigma_a(n)}{n^s} \times \sum \frac{1}{n^s}.$$

But $\{\zeta(s)\}^2 \zeta(s-a) = \sum \frac{\lambda(n)}{n_s} \times \sum \frac{n^a}{n^s}$

$$\therefore \quad \Sigma\, \lambda(d_1)\, d_2{}^a = \Sigma \sigma_a(d_r) \qquad \ldots \quad (9)$$

thus for $n = 6$, $a = 2$; the sum on each side is 66.

Lastly in the product $W = \zeta(s-a)\, \zeta(s-a-b)\, \zeta(s-a-b-c)$ $\zeta(s-a-b-c-d)$ combine the terms in pairs, in different ways.

$$\therefore \quad W = \sum \frac{n^a \sigma_b(n)}{n^s} \times \sum \frac{n^{a+b+c} \sigma_d(n)}{n^s}$$

$$= \sum \frac{n^{a+b} \sigma_c(n)}{n^s} \times \sum \frac{n^a \sigma_{b+c+d}(n)}{n^s}$$

$$= \sum \frac{n^a \sigma_{b+c}(n)}{n^s} \times \sum \frac{n^{a+b} \sigma_{c+d}(n)}{n^s}$$

$$\therefore \quad \Sigma\, d_1{}^a \sigma_b(d_1)\, d_2{}^{a+b+c} \sigma_d(d_2) = \Sigma\, d_1{}^{a+b}\, d_2{}^a\, \sigma_c(d_1)\, \sigma_{b+c+d}(d_2)$$

$$= \Sigma\, d_1{}^a\, d_2{}^{a+b} \sigma_{b+c}(d_1)\, \sigma_{c+d}(d_2). \quad (10)$$

4. If $\Sigma\, \mu(d_1) F(d_2) = f(n)$, then

$$\frac{\sum \frac{F(n)}{n^s}}{\zeta(s)} = \sum \frac{f(n)}{n^s}$$

whence $F(n) = \Sigma f(d_r)$. This is the converse of (2). Gauss' formula follows, since we know that

$$\Sigma\, d_1\, \mu(d_2) = \phi(n) \qquad \ldots \quad (11)$$

Similarly, we establish the well-known result (Kronecker : *Zahlentheorie*) if $f(n) = f(d_1)\, f(d_2)$

$$1 \div \sum \frac{f(n)}{n^s} = \sum \frac{\mu(n) f(n)}{n} \qquad \ldots \quad (12)$$

S. D. CHOWLA.

Solutions.

Question 777.

(CORRECTED). (S. KRISHNASWAMI AIYANGAR). Show that the envelope of the axes of the parabolas having double contact with the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

at the ends of a focal chord is

$$4(x^2 + y^2)^2 + ae^3(20xy^2 + 12x^3 - 12x^2 a e^3 + ay^2 e^3 + 4xa^2 e^6) = 0.$$

Solution by N. Sankara Aiyar.

The equation of a parabola having double contact with the given ellipse at the extremeties of the chord

$$y = m(x - ae)$$

is $$\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)\left(a^2 m^2 + b^2\right) = \left(y - mx + mae\right)^2$$

On writing this in the form

$$(b^2x + a^2my + k)^2 = l(a^2mx - b^2y + n)$$

and comparing co-efficients we obtain

$$k = -\frac{m^2 a^3 b^2 e(a^2 - b^2)}{a^4 m^2 + b^4}$$

Substituting this value we have for the equation of the axis :—

$$m^3 a^6 y + m^2(a^4 b^2 x - a^5 b^2 e^3) + ma^2 b^4 y + b^6 x = 0.$$

Differentiating and eliminating m we obtain the result.

Question 1269.

R. VYTHYNATHASWAMI :—Solve the equation

$$\frac{dx}{ax^2 + bx + c} = \frac{dy}{a'y^2 + b'y + c'}$$

Hence or otherwise show that the irreducible algebraic curves which lie on a given quadric and whose tangents belong to a linear complex can always be reduced to the form $\lambda^m = \mu^n$ (m prime to n). where λ, μ are the generator co-ordinates of the quadric (the point λ, μ

on the quadric is the point the parameters of the generators through which are λ, μ).

Find the invariant relation between the quadric and the complex in order that one such irreducible curve may be a twisted cubic.

Solution and Remarks by A. Narasinga Rao.

A plane in general position cuts a generator of either system in a single point; hence the generator equation of a plane section of the quadric is bilinear in λ and μ. Taking $(x_1\ x_2\ x_3\ x_4)$ to be the homogeneous co-ordinates of a point on the quadric, we have therefore equations of the form

$$\rho x_r = a_r\lambda\mu + b_r\lambda + c_r\mu + d_r \qquad (r = 1, 2, 3, 4)$$

corresponding to the sections of the quadric by the co-ordinate planes.

By replacing the x's by certain of their linear combinations, *i.e.*, by a proper choice of the tetrahedron of reference, we may reduce the above equations to

$$X_1 : X_2 : X_3 : X_4 = \lambda\mu : \lambda : \mu : 1, \qquad \dots \quad (1)$$

so that the equation of the quadric is

$$X_1 X_4 = X_2 X_3. \qquad \dots \quad (2)$$

It is clear from equation (2) that the edges (12) (13) (42) (43) form a twisted quadrilateral lying entirely on the surface.

The co-ordinates of the line joining (λ, μ) to $(\lambda + \delta\lambda, \mu + \delta\mu)$ are the 6 determinants of the matrix

$$\begin{Vmatrix} \lambda\mu & \lambda & \mu & 1 \\ \lambda d\mu + \mu d\lambda & d\lambda & d\mu & 0 \end{Vmatrix}$$

viz.,

$$-\lambda^2 d\mu,\ \lambda d\mu - \mu d\lambda,\ \mu^2 d\lambda,\ \lambda d\mu + \mu d\lambda,\ -d\lambda,\ -d\mu$$

and satisfy a linear relation if the line belongs to a linear complex. It is easy to see that on substitution we shall get a relation of the shape

$$\frac{d\lambda}{a\lambda^2 + b\lambda + c} = \frac{d\mu}{a'\mu^2 + b'\mu + c'}.^* \qquad \dots \quad (3)$$

* This may be inferred purely from *a priori* considerations since $d\lambda/d\mu$ is obviously a rational function of λ and μ which vanishes when $d\lambda = 0$, *i.e.* along the complex generators of the λ system and is infinite when $d\mu = 0$, *i.e.*

This equation takes a particularly simple form if we suppose the sides of the twisted quadrilateral spoken of above to be lines belonging to the complex. There is however, no loss of generality in this as each of the two reguli on the surface contains a pair of lines belonging to the complex, and these two pairs form a quadrilateral lying on the surface. The equation of the complex is now of the shape

$$p_{14} = kp_{23}, \qquad \ldots \quad (4)$$

and the condition that the line element $d\lambda$, $d\mu$ through λ, μ belong to the complex (4) is now

$$\lambda d\mu + \mu d\lambda + k(\lambda d\mu - \mu d\lambda) = 0,$$

or

$$\frac{d\lambda}{\lambda} = \frac{k+1}{k-1} \cdot \frac{d\mu}{\mu},$$

which gives

$$\lambda^{k-1} = \mu^{k+1} \times \text{a constant}, \qquad \ldots \quad (5)$$

as the generator equation of the complex curves on the surface. If the curves are to be algebraic, k must be a rational number and (5) takes the form

$$\lambda^m = \mu^n,$$

if the parameters of the λ or the μ system be altered by multiplication by a suitable constant.

If the equation of the quadric be taken as

$$X_1 X_4 = k' X_2 X_3, \qquad \ldots \quad (6)$$

the complex curves on (6) are given by $\lambda^m = \mu^n$ where

$$\frac{m}{n} = \frac{k+k'}{k-k'}.$$

The number $(k+k')/(k-k')$ is thus seen from purely geometrical considerations to be an absolute invariant of the quadric (6) and the

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along the complex generators of the μ system. As there are two generators of each system belonging to the complex, we have

$$\frac{d\lambda}{(\lambda-\alpha)(\lambda-\beta)} = \frac{d\mu}{(\mu-\alpha')(\mu-\beta')} \times \text{const.}$$

This gives on integration

$$\left(\frac{\lambda-\alpha}{\lambda-\beta}\right)^{\alpha'-\beta'} = \left(\frac{\mu-\alpha'}{\mu-\beta'}\right)^{\alpha-\beta} \times \text{a const.}$$

which is linearly transformable into the relation (5).

complex (4). In fact, if

$$\Sigma\, a_{rs}\, x_r\, x_s \qquad (a_{rs} = a_{sr})$$

and

$$\Sigma\, b_{rs}\, x_r\, y_s \qquad (b_{rs} + b_{sr} = 0)$$

be the equations of a quadric and a complex respectively, and we form the determinant

$$|\, a_{rs} + \lambda b_{rs} \,| = A\lambda^4 + 2B\lambda^2 + C,$$

the co-efficients A, C are invariants of the two forms separately while B is a joint invariant of both. The system has thus but one absolute invariant, namely

$$\gamma = B/(AC)^{\frac{1}{2}},$$

which, in the case of (4) and (6), works out to $(k^2 + k'^2)/2kk'$. Hence if γ be this absolute invariant, the curves on the quadric and belonging to the complex are of the type

$$\lambda^{\sqrt{\gamma+1}} = \mu^{\sqrt{\gamma-1}} \; *$$

A twisted quadrilateral in space determines a pencil of quadrics through it and also a pencil (system of two terms) of linear complexes containing the four lines. The quadric pencil contains two singular members, namely, the plane pairs while the pencil of complexes contains two which are *special*, *i.e.*, which consist of all lines which meet a fixed line. Taking the parameters of the singular members as 0 and ∞ in both cases, the quadrics are given by (6) and the complexes by (4). Two complexes whose parameters form a dyad $+k$ and $-k$ apolar to those of the singular pair are *apolar* or *reciprocal complexes*. Two quadrics whose parameters are $+k'$ and $-k'$ are *apolar* in the sense that their joint invariants θ and θ' both vanish.

When the quadric and complex are both given, the curves on the quadric and belonging to the complex are given by (5), the several curves being obtained by varying the constant on the right. If we agree to regard two curves on a quadric as *equivalent* if there exists a collineation

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* As far as one can gather from the *Encyclopedie des science Mathematiques* (Tome III, Vol. 4. Fas. 1) the geometrical significance of γ does not seem to have been noticed hitherto The significance of $\gamma = 0$ was however pointed out by Pasch (*Crelle's Journal* Vol 75) Vide also Clebsch-Lindemann *Vorlesungen über Geometrie* 2, page 343, 348.

which carries one into the other, these curves are all equivalent since the transformation $\lambda' = a\lambda$, $\mu' = \beta\mu$ which for proper values of a, β carries any curve into any other is equivalent to the collineation

$$X'_1 : X'_2 : X'_3 : X'_4 = a\beta X_1 : aX_2 : \beta X_3 : X_4.$$

Again since an interchange of λ and μ, *i.e.*, of m and n is equivalent to a change of sign of k or of k' in (5), it follows that *apolar complexes determine on the same quadric curves which are equivalent* under the collineation $\lambda' = a\mu$, $\mu' = \beta\lambda$. Correlatively *the curves determined by the same complex on apolar quadrics are seen to be equivalent.*

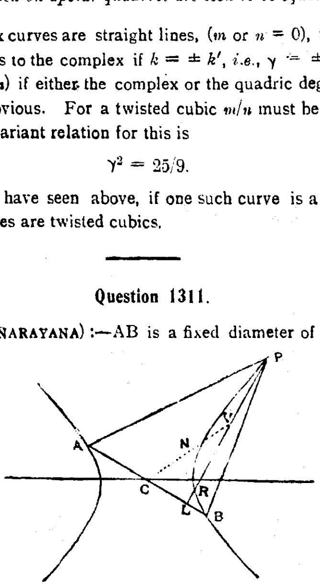
The complex curves are straight lines, (m or $n = 0$), that is, one of the reguli belongs to the complex if $k = \pm k'$, *i.e.*, $\gamma = \pm 1$. They are conics ($m = \pm n$) if either the complex or the quadric degenerates as is also otherwise obvious. For a twisted cubic m/n must be 2 or $\frac{1}{2}$ so that the necessary invariant relation for this is

$$\gamma^2 = 25/9.$$

From what we have seen above, if one such curve is a twisted cubic then all the curves are twisted cubics.

Question 1311.

(K. SATYANARAYANA) :—AB is a fixed diameter of a rectangular



hyperbola and P any point on the curve. Show that the locus of the nine-points centre of the triangle ABP is the diameter of the curve inclined to the axis at the same angle as AB.

Solutions by R. Srinivasan, Ganesan and S. Audinarayana.

Draw PL perpendicular to AB cutting the curve in R. Then R is the orthocentre of the triangle PAB. Join C, the centre, to the mid-point V of PR. Then N, the mid-point of CV is the nine-point centre of PAB. Now CV is conjugate to PR which is perpendicular to CB. Hence CV and CB are equally inclined to the axis. (Smith: *Geometrical Conics*, Chap. IV, Prop. XXVIII.) Hence the locus of N is the diameter CV.

Analytical Solutions by M. K. Kewalramani, N. M. Mehta, P. R. Venkatakrishna Iyer, V. A. Mahalingam and M. V. Seshadri.

Question 1330

(N. Durairajan) :—If I_1, I_2, I_3 ... be the centres of the inscribed circles of the triangles ABC, I_1BC, I_2BC ... ; show that I_n tends to a point I on BC, such thnt

$$BI : IC = \text{arc } BA : \text{arc } AC.$$

Solutions by R. Srinivasan, M. V. Seshadri, G. V. Krishnaswami and P. Kameswara Rao.

If $r_1, r_2, r_3 \ldots r_{n+1}$ be the in-radii of triangles ABC, I_1BC, I_2BC, ... I_nBC then

$$r_n = a \Big/ \left(\cot \frac{B}{2^n} + \cot \frac{C}{2^n}\right).$$

$\therefore \underset{n\to\infty}{\text{Lt}}\ r_n = 0$ if $0 < B, C < \pi$. *i.e.*, if the triangle be a proper one.

Hence I_n tends to a point in BC.

Again, if I_nX be the perpendicular on BC,

$$\frac{BX}{XC} = r_n \cot \frac{B}{2^n} \Big/ r_n \cot \frac{C}{2^n}$$

$$= \tan \frac{C}{2^n} \Big/ \tan \frac{B}{2^n}$$

$$\therefore \quad \frac{BI}{IC} = \underset{n\to\infty}{\text{Lt}} \frac{BX}{XC} = \underset{n\to\infty}{\text{Lt}} \tan \frac{C}{2^n} \Big/ \tan \frac{B}{2^n}$$

$$= \frac{C}{B} = \frac{\text{arc } BA}{\text{arc } AC}.$$

Question 1331.

(K. SATYANARAYANA) :—Prove that the origin lies in the acute or obtuse angle between the straight lines

$$y = m_1 x + c_1, \quad y = m_2 x + c_2$$

according as $c_1 c_2 (1 + m_1 m_2)$ is $<$ or $>$ 0.

Solutions by R. Srinivasan, H. R. Gupta, S. D. Chowla, G. V. Krishnaswamy, M. V. Seshadri, V. A. Mahalingam and P. Kameswara Rao.

The two lines meet at a point P whose co-ordinates are

$$\left(\frac{c_2 - c_1}{m_1 - m_2}, \; \frac{m_1 c_2 - m_2 c_1}{m_1 - m_2} \right)$$

Let the Y-axis cut the lines in Q and R respectively.

Then O lies in the acute angle, if

(1) OQ and OR are opposite in sign and $Q\hat{P}R$ is acute,

i.e.,
$$QR^2 < PQ^2 + PR^2;$$

or (2) OQ and OR are of the same sign and $Q\hat{P}R$ is obtuse,

i.e.
$$QR^2 > PQ^2 + PR^2.$$

Now $QR^2 \gtrless PQ^2 + PR^2$ according as

$$\left(c_1 - c_2\right)^2 \gtrless \left(1 + m_1^2\right) \left(\frac{c_2 - c_1}{m_1 - m_2}\right)^2 + \left(1 + m_2^2\right) \left(\frac{c_2 - c_1}{m_1 - m_2}\right)^2$$

i.e., $(m_1 - m_2)^2 \gtrless m_1^2 + m_2^2 + 2$

i.e., as $2(1 + m_1 m_2) \lessgtr 0$

i.e., as $1 + m_1 m_2$ is negative or positive

∴ if O lies in the acute angle formed by the two lines, c_1 and c_2 must be of the same sign or different signs according as $1 + m_1 m_2$ is negative or positive, that is, $c_1 c_2 (1 + m_1 m_2)$ is negative. Hence the result.

...e origin to any other point x, y we deduce that the ...he acute or obtuse angle between the lines accord-

$$...n_2 (y - m_1 x - c_1)(y - m_2 x - c_2) \text{ is } < \text{ or } > 0.$$

Question 1367.

(S. D. CHOWLA) :—Prove that

$$e^{-\beta^2} - e^{-(5\beta)^2} - e^{-(7\beta)^2} + e^{-(11\beta)^2} + e^{-(13\beta)^2} - - + + \ldots$$

$$= \frac{\sqrt{\pi}}{2\sqrt{3}\beta} \{ e^{-(\pi/12\beta)^2} - e^{-(5\pi/12\beta)^2} - e^{-(7\pi/12\beta)^2} + + - - \ldots$$

Solution by the Proposer.

It is a well-known result in Cauchy's Theory of Reciprocal Func- ...at if :

$$\int_0^\infty \phi(x) \cos knx \, dx = \psi(n)$$

$$\phi(\alpha) - \phi(5\alpha) - \phi(7\alpha) + \phi(11\alpha) + \phi(13\alpha) - - + + \ldots \}$$
$$(\beta) - \psi(5\beta) - \psi(7\beta) + \psi(11\beta) + \psi(13\beta) - - + + \ldots$$

...he ...ion

$$\alpha\beta = \frac{\pi}{6k} \quad \ldots \quad \ldots \quad \ldots (1)$$

But we know that

$$e^{-n^2} = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-x^2} \cos 2nx \, dx \quad \ldots \quad \ldots (2)$$

Here $\psi(n) = \frac{\sqrt{\pi}}{2} e^{-n^2}$, $\phi(n) = e^{-n^2}$ $k = 2$, so that (1) gives

$$e^{-\beta^2} - e^{-(5\beta)^2} - e^{-(7\beta)^2} + e^{-(11\beta)^2} + e^{-(13\beta)^2} - - + + \ldots$$

$$= \sqrt{3}.\frac{\pi}{12\beta}\frac{2}{\sqrt{\pi}} \left\{ e^{-(\pi/12\beta)^2} - e^{-(5\pi/12\beta)^2} - e^{-(7\pi/12\beta)^2} + + - - \right\}$$

We notice that the result is obviously correct, when $\beta^2 = \frac{\pi}{12}$.

Questions for ...

Proposers of Questions are requested to send t...
along with their questions.

1462. (V. RAMASWAMY AIYAR):—Given a trian... variable Simson line cut the sides in A', B', C', then th... ocus ... mean centre of A', B', C' for any given system of multiples (w... sum is not zero) is an ellipse having triple contact with the Stein... tricusp.

More generally, if a variable tangent to a three-cusped hypocycl... cut n fixed tangents in P_1, P_2 ... P_n, the locus of the mean centre o... P_1, P_2, ... P_n for any given system of multiples (whose sum is not zero) is an ellipse having triple contact with the tricusp.

1463. (V. RAMASWAMI AIYAR):—Prove that for every orthocentric tetrahedron ABCD there is another $A'B'C'D'$, such that O, O' being the ortho-centres of these tetrahedra, we shall have

$$OA = O'A', \; OB = O'B', \; OC = O'C', \; OD = O'D'.$$

Show that if the polar sphere of the one is real that of the ... imaginary and *vice versa*. Prove also that the circum-radii of ... tetrahedra are equal.

1464. (R. VAIDYANATHASWAMY):—x and y are e... functions of z, defined by the respective irreducible equations

$$F_1(x, z) = 0$$

$$F_2(y, z) = 0,$$

where F_1 is of the mth degree and F_2 of the nth degree in z. ... x and y have the same branch points, shew that the z-resultants o... F_1 and F_2 will resolve into two factors, each of which is of the nt... degree in x and mth degree in y.

Shew also that this is the only possible way in which the resultan... can resolve.

1465. (K. V. VEDANTAM):—If the focus of an in-conic ... triangle lies on the polar circle its directrix passes through the ... centre.

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...URNALS RECEIVED

...nuary and February 1927.

...Journals.

...Mathematica, **49**, 3 & 4.
...merican Journal of Mathematics, **48**, 4 (2 copies).
...strophysical Journal, **64**, 3 & 4.
...ulletin des Sciences Mathematiques, November 1926.
...ulletin of the American Mathematical Society, **32**, 5.
...apanese Journal of Mathematics, **3**. 1 & 2.
Journal für die Reine und ang. Math , **157**, 2.
Liouville's Journal, Tome **5**, 4.
Mathematische Annalen, **97**, 1 & 2.
...lathematical Gazette, **13**, 185 (2 copies).
...ssenger of Mathematics, **56**, 6 & 7 (2 copies).
...hly Notices of the Royal Astronomical Society, **87**, 2.
... **118**, 2975 to 2982.
...ophical Magazine, **3**, 13.
...ar Astronomy, **34**, 10 and **35**, 1.
...edings of the Edinburgh Mathematical Society, **44**, 2.
...edings of the Royal Society, **113**, 765.
...nti del Cirolo Mathematico di Palermo, **50**, 11.
...ical Journal, **27**, 3 & 4.
...f the American Mathematical Society, **28**, 4.

Books.

...sity Calendar, 1926-1927, Vols. 1 & 2,
...No. 21 (Theory of Integration, by L. C. Young)
...es Annalytiques Entre deux Plans Pro...
...yl glyoximm, 2 & 3.

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